$$u = -a_{12}\frac{qR}{h}\alpha, \qquad v = -a_{26}\frac{qR}{h}\alpha, \qquad w = -a_{22}\frac{qR^2}{h} \tag{6.10}$$

The values of the displacements (6.10) presented here agree with the appropriate displacements obtained by classical theory [1], just as it should. Assuming $a_{13} = a_{23} = a_{36} = 0$, we pointedly neglect the normal stresses $\sigma_\gamma$ while determining the displacements, and in combination with the initial assumptions we accepted above; this forms the complex of initial hypotheses of the classical theory.

Finally, let us mention that the weakened membrane state conditions (5.6) are satisfied in the example discussed $(p = 0,\ q \neq 0)$ if, for example $a_{13} / a_{12} < 1$, $a_{23} / a_{22} < 1$, $a_{36} / a_{26} < 1$ (for $m = 1$).

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Translated by M. D. F.

# ON ELASTICITY RELATIONSHIPS IN THE LINEAR THEORY OF THIN ELASTIC SHELLS

PMM Vol. 34, №6, 1970, pp. 1136-1138

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(Received February 18, 1970)

Questions of constructing tensor elsticity relationships within the accuracy of the Kirchhoff-Love hypotheses are discussed herein. It is clarified that it is impossible to conserve simultaneously the static-geometric analogy and to assure application of theorems of the theory of elasticity by using not too complex elasticity relationships. In this connection, two modifications of the elasticity relationships in the linear theory of thin elastic shells are proposed. The first modification retains these theorems in the linear theory of thin elastic shells. The second modification satisfies the requirements of the static-geometric analogy, but violates the reciprocity theorem (in the small).

Among the possible modifications of the elasticity relationships used in the linear theory of thin elastic shells, one of the most simple ones is the modification presented by the authors of [1] and [2]. Nevertheless, these relationships answer a number of requirements to be discussed below. At the same time the authors of [3] indicated that these relationships are not of tensor character.

In this paper we consider the construction of tensor elasticity relationships differing slightly in the lines of curvature from relationships presented in [1] and [2].

**1.** Let us use the following notation: $a_{mn}$, $b_{mn}$ are tensors of the first and second quadratic forms, $a$ is the fundamental determinant and $c_{mn}$ is the auxiliary antisymmetric tensor whose components are $c_{nn} = 0$, $c_{12} = -c_{21} = a^{1/2}$.

The remaining notation differs from the notation in the monograph [4] only in that $S_{21}$, $-S_{12}$, $H_{21}$, $-H_{12}$, $R_{11}$, $R_{22}$ are utilized instead of $S_1$, $S_2$, $H_1$, $H_2$, $R_1$, $R_2$ respectively.

**2.** The known modification of the elasticity relationships (see [1] and [2])

$$T_i = B(\varepsilon_i + \sigma\varepsilon_j), \quad S_{ij} = B(1-\sigma)\left(\frac{\omega}{2} + \frac{h^2}{3}\frac{\tau}{R_{ii}}\right) \quad \left(B = \frac{2Eh}{1-\sigma^2}\right) \tag{2.1}$$

$$G_i = -D(\varkappa_i + \sigma\varkappa_j), \quad H_{12} = H_{21} = D(1-\sigma)\tau, \quad i \neq j = 1,2 \quad \left(D = \frac{2Eh^3}{3(1-\sigma^2)}\right)$$

possesses a number of advantages:

a) The elasticity relationships do not contradict the sixth equation of equilibrium;

b) The stress resultants and moments are representable as a linear transformation of the strain energy components with the symmetric matrix of coefficients (the strain energy components are determined by the expressions [4]

$$\varepsilon^{(i)} = \varepsilon_i, \quad \varkappa^{(i)} = \varkappa_i, \quad \omega^{(1)} = -\omega^{(2)} = {}^1\!/_2\omega, \quad \tau^{(i)} = \tau - {}^1\!/_2\omega / R_{ii} \quad (i = 1,2)\ );$$

c) The elastic potential is not negative for any values of the strain components;

d) The relationships (2.1), referred to the strain energy components, satisfy the conditions of the static-geometric analogy.

The conditions (a)–(c) permit to transfer the fundamental theorems of the theory of elasticity to the theory of shells. Condition (d) can be treated as the condition of applicability of the complex method given in [5] to this modification of shell theory.

The static-geometric analogy is that the shell theory equations can be separated into two groups, which go into one another when taking account of the relationships [4]

$$\{T_i, S_i, G_i, H_i\} \leftrightarrow \{-\varkappa^{(j)}, \tau^{(j)}, \varepsilon^{(j)}, -\omega^{(j)}\}, \qquad i \neq j = 1,2$$

$$\{\sigma, B, D\} \leftrightarrow \{-\sigma, -D', -B'\}, \qquad B' = \frac{1}{2Eh}, \qquad D' = \frac{3}{2Eh^3}$$

**3.** The authors of [3] have shown that the elasticity relationships of [1] and [2] are not of tensor character. It is hence natural to pose the question of altering these relationships so that they would remain sufficiently simple when given tensor form (this is understood to be the requirement that the dependence on the curvature tensor be linear). Moreover, compliance with conditions (a)–(d) in Sect. 2 must be conserved.

Let us introduce the first and second strain tensors $\varepsilon_{mn}$ and $\mu_{mn}$ [6]. If it is agreed to understand $[d_{ik}]$ to be the physical components of the tensor $d_{ik}$ at the lines of curvature, then $\varepsilon_{mn}$ and $\mu_{mn}$ will be understood to be the tensors for which

$$[\varepsilon_{ii}] = \varepsilon_i, \quad [\varepsilon_{ij}] = {}^1\!/_2\omega, \quad [\mu_{ii}] = \varkappa_i, \quad [\mu_{ij}] = \tau - {}^1\!/_2\omega / R_{jj}, \qquad i \neq j = 1,2$$

Let us also introduce the tensor of the tangential forces $T^{mn}$ and the moment tensor $M^{mn}$ by considering them to satisfy the requirements

$$[T^{ii}] = T_i, \quad [T^{ij}] = S_{ij}, \quad [M^{ii}] = G_i, \quad [M^{ij}] = H_{ij}, \quad i \neq j = 1,2$$

The formulas connecting the moment and tangential force tensors with the first and second strain tensors can be written in the general case as

$$T^{mn} = BE^{mn\alpha\beta}\varepsilon_{\alpha\beta} + DF^{mn\alpha\beta}\mu_{\alpha\beta}, \qquad M^{mn} = DG^{mn\alpha\beta}\mu_{\alpha\beta} + DH^{mn\alpha\beta}\varepsilon_{\alpha\beta}$$

Let us consider the tensors $E$ and $G$ to be selected so that

$$E^{mn\alpha\beta} = G^{mn\alpha\beta} = a^{m\alpha}a^{n\beta} + \sigma c^{n\alpha}c^{m\beta} \tag{3.1}$$

Moreover, if we set $F = H = 0$, then tensor elasticity relationships are obtained which at the lines of curvature have the form

$$T_i = B(\varepsilon_i + \sigma\varepsilon_j), \qquad S_{ij} = {}^1/_2 B(1-\sigma)\omega$$

$$G_i = -D(\varkappa_i + \sigma\varkappa_j), \quad H_{ij} = D(1-\sigma)(\tau - {}^1/_2\omega/R_{jj}) \qquad i \neq j = 1,2$$

To the accuracy of terms in ${}^1/_2\omega/R_{ij}$, which are needed to give the elasticity relationships tensor form, this is the simplest modification of the Love elasticity relationships. It does not satisfy the requirements of Sect. 2.

Retaining the formulas (3.1) for $E$ and $G$, let us seek such $F$ and $H$ for which the requirements in Sect. 2 would be satisfied, at least partially, by assuming that these tensors are comprised of the tensors $a$, $b$ and $c$, and are linear in $b$.

**4.** It can be shown that in tensor notation it is impossible to satisfy simultaneously the symmetry condition of the matrix of coefficients in the elasticity relationships and the requirement of the static-geometric analogy. This means that no tensor analog of the relationships presented in [1] and [2] exist.

If the requirement to retain the static-geometric analogy is discarded, then elasticity relationships satisfying all the remaining requirements exist, and can be taken in the form

$$T^{mn} = B(a^{m\alpha}a^{n\beta} + \sigma c^{n\alpha}c^{m\beta})\varepsilon_{\alpha\beta} - D(1-\sigma)b^{n\alpha}a^{m\beta}\mu_{\alpha\beta}$$

$$M^{mn} = D(a^{m\alpha}a^{n\beta} + \sigma c^{n\alpha}c^{m\beta})\mu_{\alpha\beta} - D(1-\sigma)b^{\beta n}a^{\alpha m}\varepsilon_{\alpha\beta}$$

At the lines of curvature they are (4.1)

$$T_i = B(\varepsilon_i + \sigma\varepsilon_j) + \left\{D(1-\sigma)\frac{\varkappa_i}{R_{ii}}\right\} \qquad S_{ij} = (1-\sigma)\left[B\frac{\omega}{2} + \frac{D}{R_{jj}}\left(\tau - \left\{\frac{\omega}{2R_{ii}}\right\}\right)\right]$$

$$G_i = -D[\varkappa_i + \sigma\varkappa_j + \{(1-\sigma)\varepsilon_i/R_{ii}\}], \qquad H_{12} = H_{21} = D(1-\sigma)\tau \quad (i \neq j = 1,2)$$

Terms missing from the relationships presented in [1] and [2] are placed in braces. It is easy to verify the nonnegativity of the elastic potential for (4.1)

$$2W = T^{mn}\varepsilon_{mn} + M^{mn}\mu_{mn} = B\{\varepsilon_1^2 + \varepsilon_2^2 + \nu_1^2 + \nu_2^2 + 2\sigma(\varepsilon_1\varepsilon_2 + \nu_1\nu_2) +$$

$$+ \frac{2h}{\sqrt{3}}\left(\frac{\nu_1\varepsilon_1}{R_{11}} + \frac{\nu_2\varepsilon_2}{R_{22}}\right)(1-\sigma) + (1-\sigma)\left[\left(1 - \frac{h^2}{R_{11}R_{22}}\right)\frac{\omega^2}{2} + \frac{2}{3}h^2\tau^2\right]\Bigg\} =$$

$$= B\{[{}^1/_2\varepsilon_1^2 + {}^1/_2\varepsilon_2^2 + 2\sigma\varepsilon_1\varepsilon_2] + [{}^1/_2\nu_1^2 + {}^1/_2\nu_2^2 + 2\sigma\nu_1\nu_2] +$$

$$+ \left[\frac{1}{2}\varepsilon_1^2 + \frac{1}{2}\nu_1^2 + \frac{2\sigma h}{\sqrt{3}R_{11}}\nu_1\varepsilon_1\right] + \left[\frac{1}{2}\varepsilon_2^2 + \frac{1}{2}\nu_2^2 + \frac{2\sigma h}{\sqrt{3}R_{22}}\nu_2\varepsilon_2\right] +$$

$$+ (1-\sigma)[{}^1/_2(1 - {}^1/_3 h^2/R_{11}R_{22})\omega^2 + {}^2/_3 h^2\tau^2]\} \geqslant 0$$

$$(\nu_i = \varkappa_i h / \sqrt{3}, \; i = 1, 2)$$

since the terms included in each pair of square brackets are nonnegative under the condition $\sigma \leqslant {}^1/_2$.

**5.** Conversely, the static-geometric analogy can be retained and the symmetry of the matrix of coefficients in the elasticity relationships violated. In this case the reciprocity theorem is violated, and this can be justified by the fact that conservation of the static-geometric analogy opens the way to application of the complex method.

Then the elasticity relationships satisfying all the remaining requirements, including the condition of nonnegativity of the elastic potential for $\sigma \leqslant 1/2$, can be taken in the form

$$T^{mn} = B\,(a^{n\alpha}a^{m\beta} + \sigma c^{m\alpha}c^{n\beta})\,\varepsilon_{\alpha\beta} + D\,(1-\sigma)\,c^{ns}b_{sk}c^{k\alpha}a^{m\beta}\mu_{\alpha\beta}$$

$$\mu^{mn} = D'\,(a^{n\alpha}a^{m\beta} - \sigma c^{m\alpha}c^{n\beta})\,M_{\alpha\beta} + B'\,(1+\sigma)\,b^{n\alpha}a^{m\beta}T_{\alpha\beta}$$

At the lines of curvature the following elasticity relationships correspond to them:

$$T_i = B\,(\varepsilon_i + \sigma\varepsilon_j) - D\,(1-\sigma)\varkappa_i\,/\,R_{jj}$$

$$S_{ij} = (1-\sigma)\left[B\frac{\omega}{2} - \frac{D}{R_{ii}}\left(\tau - \frac{\omega}{2R_{ii}}\right)\right]$$

$$\varkappa_i = -D'\,(G_i - \sigma G_j) + B'\,(1+\sigma)\,T_i/R_{ii}$$

$$\tau - \frac{\omega}{2R_{ii}} = (1+\sigma)\left[D'H_{ij} - B'\frac{S_{ij}}{R_{ii}}\right], \qquad i \neq j = 1,2$$

These elasticity relationships in a general tensor description are much simpler than the relationships of Lur'e, which do not, moreover, satisfy the requirement of linearity in the tensor $b$ and the static-geometric analogy.

The authors of [3] showed that a shell theory satisfying all the requirements in Sect. 2 can be formally constructed. But to do this, they had to replace the stress resultants and strains in the shell theory equations by their linear combinations, which detriments the physical clearness of the shell theory equations.

The author is grateful to A. L. Gol'denveizer for continued attention to this research.

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Translated by M. D. F.